Consider a localised time dependant distribution of currents as source. The total magnetic field produced by the distribution is the sum of the *near zone* (or *static*) field, *intermediate zone* (or *induction*) field and *far zone* (or *radiation*) field (see Jackson, section 9.1) :

$$\mathbf{B}_{tot}(t,\mathbf{r}) = \mathbf{B}_{near}(t,\mathbf{r}) + \mathbf{B}_{inter}(t,\mathbf{r}) + \mathbf{B}_{far}(t,\mathbf{r}). \qquad \textbf{(equ. 1)}$$

In the case of a rotating magnetic dipole, we have (we are omitting the factor $\mu_0/4\pi$ for the sake of simplicity) :

$$\mathbf{B}_{near}(t,\mathbf{r}) = \frac{3}{r^5}(\boldsymbol{\mu}\cdot\mathbf{r})\,\mathbf{r} - \frac{1}{r^3}\,\boldsymbol{\mu}, \qquad \textbf{(equ. 2)}$$

$$\mathbf{B}_{inter}(t,\mathbf{r}) = \frac{3}{cr^4}(\dot{\boldsymbol{\mu}}\cdot\mathbf{r})\,\mathbf{r} - \frac{1}{cr^2}\,\dot{\boldsymbol{\mu}}, \qquad \textbf{(equ. 3)}$$

$$\mathbf{B}_{far}(t,\mathbf{r}) = \frac{1}{c^2r^3}(\ddot{\boldsymbol{\mu}}\cdot\mathbf{r})\,\mathbf{r} - \frac{1}{c^2r}\,\ddot{\boldsymbol{\mu}} \equiv \frac{1}{c^2r^3}\,\mathbf{r}\times(\mathbf{r}\times\ddot{\boldsymbol{\mu}}), \qquad \textbf{(equ. 4)}$$

where the dot means a time derivative. The magnetic moment $\boldsymbol{\mu}$ is steadily rotating around the $\hat{\mathbf{z}}$ axis with an inclination angle α and angular velocity ω. Since the fields must be evaluated at the retarded time, we have (take note that a common constant μ was factored out) :

$$\boldsymbol{\mu}(t,\mathbf{r}) = \cos(\omega t - k(r-R))\sin\alpha\,\hat{\mathbf{x}} + \sin(\omega t - k(r-R))\sin\alpha\,\hat{\mathbf{y}} + \cos\alpha\,\hat{\mathbf{z}}, \qquad \textbf{(equ. 5)}$$

where $k = \omega/c$ is the *wave number* and R is the source radius, assumed to be a uniformly magnetized rotating sphere. Although the equations above are sufficient to build the magnetic field models with ***Mathematica***, it is useful to introduce the following two *locally* orthogonal unit vectors :

$$\mathbf{u}(\mathbf{r}) = \cos(k(r-R))\,\hat{\mathbf{x}} - \sin(k(r-R))\,\hat{\mathbf{y}}, \qquad \textbf{(equ. 6)}$$

$$\mathbf{v}(\mathbf{r}) = \sin(k(r-R))\,\hat{\mathbf{x}} + \cos(k(r-R))\,\hat{\mathbf{y}}. \qquad \textbf{(equ. 7)}$$

Equ. 5 becomes

$$\boldsymbol{\mu}(t,\mathbf{r}) = \sin\alpha\cos(\omega t)\,\mathbf{u} + \sin\alpha\sin(\omega t)\,\mathbf{v} + \cos\alpha\,\hat{\mathbf{z}} \qquad \text{(equ. 8)}$$

and the total magnetic field can now be expressed as follows :

$$\mathbf{B}(t,\mathbf{r},\alpha) = \mathbf{F}(\mathbf{r})\sin\alpha\cos(\omega t) + \mathbf{G}(\mathbf{r})\sin\alpha\sin(\omega t) + \mathbf{H}(\mathbf{r})\cos\alpha, \qquad \textbf{(equ. 9)}$$

where :

$$\mathbf{F}(\mathbf{r}) = \left(\frac{3}{r^5}(\mathbf{u}\cdot\mathbf{r})\,\mathbf{r} - \frac{1}{r^3}\,\mathbf{u}\right) + k\left(\frac{3}{r^4}(\mathbf{v}\cdot\mathbf{r})\,\mathbf{r} - \frac{1}{r^2}\,\mathbf{v}\right) - k^2\left(\frac{1}{r^3}\,\mathbf{r}\times(\mathbf{r}\times\mathbf{u})\right), \qquad \textbf{(equ. 10)}$$

$$\mathbf{G}(\mathbf{r}) = \left(\frac{3}{r^5}(\mathbf{v}\cdot\mathbf{r})\,\mathbf{r} - \frac{1}{r^3}\,\mathbf{v}\right) - k\left(\frac{3}{r^4}(\mathbf{u}\cdot\mathbf{r})\,\mathbf{r} - \frac{1}{r^2}\,\mathbf{u}\right) - k^2\left(\frac{1}{r^3}\,\mathbf{r}\times(\mathbf{r}\times\mathbf{v})\right), \qquad \textbf{(equ. 11)}$$

$$\mathbf{H}(\mathbf{r}) = \frac{3}{r^5}(\hat{\mathbf{z}}\cdot\mathbf{r})\,\mathbf{r} - \frac{1}{r^3}\,\hat{\mathbf{z}}. \qquad \textbf{(equ. 12)}$$

Notice that the vectors $\mathbf{F}(\mathbf{r})$, $\mathbf{G}(\mathbf{r})$, and $\mathbf{H}(\mathbf{r})$ don't form an orthogonal set. Remarkably, the time evolution of the magnetic field is such that all the field lines are *globally* rotating like a rigid body around the center $\mathbf{r} = 0$, despite the fact that there are waves which are radiated away in this field ! This very surprising result can be proved as follows. If it's true that the total magnetic field simply rotates like a rigid body, then the following vectorial equation should be satisfied, for any point in space :

$$\mathbf{B}(t, \mathbf{r}) = \mathcal{R}(\hat{\mathbf{z}}, \omega t)\, \mathbf{B}(0, \mathbf{r}'), \qquad \text{(equ. 13)}$$

where $\mathcal{R}(\hat{\mathbf{z}}, \omega t)$ is a time dependant rotation operator applied to the initial magnetic field evaluated at the "previous" position : $\mathbf{r}' = \mathcal{R}^{-1}(\hat{\mathbf{z}}, \omega t)\, \mathbf{r}$ (inverse rotation applied on the position vector). That is, the initial field vector is *locally* rotated after being parallel transported to another point. Equ. 13 is thus an algebraic constraint on the time evolution of the field (rigid rotation). Explicitely :

$$\mathcal{R}(\hat{\mathbf{z}}, \omega t)\, \mathbf{B} = \mathbf{B} \cos(\omega t) + \hat{\mathbf{z}} \times \mathbf{B} \sin(\omega t) + \hat{\mathbf{z}}(\hat{\mathbf{z}} \cdot \mathbf{B})\big(1 - \cos(\omega t)\big), \qquad \text{(equ. 14)}$$

$$\mathbf{r}' = \mathbf{r} \cos(\omega t) - \hat{\mathbf{z}} \times \mathbf{r} \sin(\omega t) + \hat{\mathbf{z}}(\hat{\mathbf{z}} \cdot \mathbf{r})\big(1 - \cos(\omega t)\big). \qquad \text{(equ. 15)}$$

Now, in the simpler case of $\alpha = 90°$, the field (equ. 9) has the following form :

$$\mathbf{B}(t, \mathbf{r}, 90°) = \mathbf{F}(\mathbf{r}) \cos(\omega t) + \mathbf{G}(\mathbf{r}) \sin(\omega t).$$

It isn't obvious at all that this is a rigid evolution of the field, since the vectors $\mathbf{F}$ and $\mathbf{G}$ are complicated expressions (equs. 10 and 11). However, they do obey the algebraic constraint equ. 13 :

$$\mathbf{F}(\mathbf{r}) \cos(\omega t) + \mathbf{G}(\mathbf{r}) \sin(\omega t) \equiv \mathcal{R}(\hat{\mathbf{z}}, \omega t)\, \mathbf{F}(\mathbf{r}').$$

The generalization to an angle $\alpha \neq 90°$ is almost trivial. It's easy to verify that $\mathbf{H}(\mathbf{r})$ is invariant under the rotation (actually, this is the field of the aligned static dipole associated to $\alpha = 0$) : $\mathcal{R}(\hat{\mathbf{z}}, \omega t)\, \mathbf{H}(\mathbf{r}') \equiv \mathbf{H}(\mathbf{r})$. So, using the previous result from the case $\alpha = 90°$, we get :

$$\mathbf{B}(t, \mathbf{r}) = \big(\mathbf{F}(\mathbf{r}) \cos(\omega t) + \mathbf{G}(\mathbf{r}) \sin(\omega t)\big) \sin\alpha + \mathbf{H}(\mathbf{r}) \cos\alpha$$

$$\equiv \mathcal{R}(\hat{\mathbf{z}}, \omega t)\big(\mathbf{F}(\mathbf{r}') \sin\alpha + \mathbf{H}(\mathbf{r}') \cos\alpha\big).$$

This result is physically amazing since there's radiation flowing in the magnetic field, while all the field lines are rotating like a rigid body !

The time dependance of the magnetic field implies the production of an induced electric field. Maxwell's equations in vacuum give

$$\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}. \qquad \text{(equ. 16)}$$

Using equs. 1-4, we find (see also Jackson, section 9.3) :

$$\mathbf{E}(t,\mathbf{r}) = \frac{1}{r^3}\,\mathbf{r}\times\dot{\boldsymbol{\mu}} + \frac{1}{cr^2}\,\mathbf{r}\times\ddot{\boldsymbol{\mu}}. \qquad \textbf{(equ. 17)}$$

Equivalently, using equs. 9-12 we find (or we can simply put equ. 8 into equ. 17) :

$$\mathbf{E}(t,\mathbf{r},\alpha) = \mathbf{P}(\mathbf{r})\,c\,\sin\alpha\cos(\omega t) + \mathbf{Q}(\mathbf{r})\,c\,\sin\alpha\sin(\omega t), \qquad \textbf{(equ. 18)}$$

where :

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r^3}\,\mathbf{r}\times\mathbf{v} - \frac{k^2}{r^2}\,\mathbf{r}\times\mathbf{u}, \qquad \textbf{(equ. 19)}$$

$$\mathbf{Q}(\mathbf{r}) = -\frac{k}{r^3}\,\mathbf{r}\times\mathbf{u} - \frac{k^2}{r^2}\,\mathbf{r}\times\mathbf{v}. \qquad \textbf{(equ. 20)}$$

The induced electric field is also rotating like a rigid body :

$$\mathbf{E}(t,\mathbf{r}) = \mathcal{R}(\hat{\mathbf{z}},\omega t)\,\mathbf{E}(0,\mathbf{r}'). \qquad \text{(equ. 21)}$$

Since the rotating dipole is emitting electromagnetic waves, it slowly loses energy and angular momentum. The angular velocity ω can't be a true constant, unless there is some external intervention to maintain the rotation state (falling matter from an accretion disk, for example). The Poynting vector $\wp = \mathbf{E}\times\mathbf{B}$, integrated on the surface of a very large sphere ($r\to\infty$) centered on the source, gives the *total power radiated away* by the rotating dipole. Some calculus gives (we now reintroduce the factor $\mu_0\mu/4\pi$ which was omitted from the fields) :

$$P = \oint \wp\cdot d\mathbf{S} = \frac{\mu_0\mu^2}{6\pi c^3}\,\omega^4\sin^2\alpha. \qquad \text{(equ. 22)}$$

We can give a rough estimate of the rotation damping, if we assume a spherical body of rotational energy $E = \frac{1}{2}I\omega^2$ and moment of inertia $I = \frac{2}{5}MR^2$. Equating the rate of energy lost with the radiated power gives

$$\frac{dE}{dt} = I\omega\,\frac{d\omega}{dt} = -\frac{\mu_0\mu^2}{6\pi c^3}\,\omega^4\sin^2\alpha.$$

Hence $\frac{d\omega}{dt} = -a\,\omega^3$ and

$$\omega(t) = \frac{\omega_0}{\sqrt{1+2a\omega_0^2 t}}. \qquad \text{(equ. 23)}$$

For a pulsar of radius $R = 10\,\text{km}$, mass $M = 1.4\,M_\odot$, inclination angle $\alpha = 90°$ (worst case scenario), initial rotation period $T_0 = 0.001\,\text{sec}$, and equatorial magnetic field strength $B_{surf}\sim 10^{10}\,\text{T}$, we get $2a\omega_0^2 \approx 2\times10^{-5}\,\text{sec}^{-1}$. This implies a strong angular velocity decrease with time. For $T_0 = 0.01\,\text{sec}$ and $B_{surf}\sim 10^{8}\,\text{T}$, we get $2a\omega_0^2 \approx 2\times10^{-11}\,\text{sec}^{-1}$, which is a slow decay rate. For $T_0 = 1\,\text{sec}$ and $B_{surf}\sim 10^{7}\,\text{T}$, we get $2a\omega_0^2 \approx 2\times10^{-17}\,\text{sec}^{-1}$, which corresponds to a period doubling of about 2 billion years !

Next, we want to numerically solve the relativistic version of Newton's equation for a " *test* " particle of charge q and proper mass m_0 moving in the electromagnetic field, taking into account the gravitational force and the special relativistic effects. Newton's equation in any inertial reference frame is :

$$\frac{d\mathbf{p}}{dt} = \mathbf{F}_{tot}, \qquad \text{(equ. 24)}$$

where $\mathbf{p} = \gamma m_0 \mathbf{v}$ is the relativistic linear momentum of the particle and $\mathbf{F}_{tot} = \mathbf{F}_{grav} + \mathbf{F}_{ele} + \mathbf{F}_{magn}$ is the total force acting on the particle. Explicitely :

$$\mathbf{F}_{grav} = -\frac{GMm_0}{r^3}\,\mathbf{r}, \qquad \mathbf{F}_{ele} = q\,\mathbf{E}(t,\mathbf{r}), \qquad \mathbf{F}_{magn} = q\,\mathbf{v}\times\mathbf{B}(t,\mathbf{r}). \qquad \text{(equ. 25)}$$

The presence of the relativistic factor $\gamma = (1 - v^2/c^2)^{-1/2}$ in the derivative of equ. 24 may cause a problem for the numerical integration. It's easy to verify that equ. 24 is equivalent to the following equation (from now on, we will absorb the constant c into the definition of the particle's velocity) :

$$\gamma m_0 c\,\frac{d\mathbf{v}}{dt} = \mathbf{F}_{tot} - (\mathbf{v}\cdot\mathbf{F}_{tot})\,\mathbf{v}, \qquad \text{(equ. 26)}$$

Since the goal is to build a 3D model of the particle's trajectory and to show it in ***Celestia***, it's preferable to define the trajectory in the rotating frame centered on the source, while using the time t read on the inertial clock. Thus (remember that $\boldsymbol{\omega} = kc\,\hat{\mathbf{z}}$) :

$$\mathbf{v} = \mathbf{v}' + k\,\hat{\mathbf{z}}\times\mathbf{r}, \qquad \text{(equ. 27)}$$

$$\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{v}'}{dt} + 2ck\,\hat{\mathbf{z}}\times\mathbf{v}' + ck^2\,\hat{\mathbf{z}}\times(\hat{\mathbf{z}}\times\mathbf{r}), \qquad \text{(equ. 28)}$$

where $\mathbf{v}'$ is the particle's velocity in the rotating frame. The last two terms of equ. 28 are associated to the Coriolis and centrifugal forces, respectively. Next, to allow a proper numerical integration, we need to normalize the position and time coordinates to dimensionless quantities. Lets introduce an arbitrary radius R as length unit (the source radius) and $T = R/c$ as time unit, such that $\mathbf{r}_* = \mathbf{r}/R$ and $t_* = t/T$ are now dimensionless. Since the velocity was normalised with the c constant, it's already dimensionless : $\mathbf{v} \equiv \mathbf{v}_*$. The equation of motion (equ. 26) then becomes

$$\frac{d\mathbf{v}_*'}{dt_*} = \frac{1}{\gamma}\left(\mathbf{F}_* - (\mathbf{v}_*\cdot\mathbf{F}_*)\,\mathbf{v}_*\right) - 2k_*\,\hat{\mathbf{z}}\times\mathbf{v}_*' - k_*^2\,\hat{\mathbf{z}}\times(\hat{\mathbf{z}}\times\mathbf{r}_*) \qquad \textbf{(equ. 29)}$$

where

$$\mathbf{F}_* = \frac{R}{m_0 c^2}\,\mathbf{F}_{tot} = -\,\alpha_{grav}\,\frac{1}{r_*^3}\,\mathbf{r}_* + \alpha_{\mathrm{EM}}(\mathbf{E}_* + \mathbf{v}_*\times\mathbf{B}_*) \qquad \textbf{(equ. 30)}$$

is the normalised (dimensionless) force and $k_* = kR \equiv \omega_*$ is the dimensionless angular velocity. Of course, the field components $\mathbf{E}_*$ and $\mathbf{B}_*$ must now be evaluated at time $t_* = 0$ in the rotating frame, since they are rigidly rotating with the same angular velocity. The dimensionless coupling constants are

$$\alpha_{grav} = \frac{GM}{Rc^2}, \qquad \alpha_{\mathrm{EM}} = \frac{qR}{m_0 c}\,B_{surf}. \qquad \textbf{(equ. 31)}$$

Equs. 29-30 may now be numerically integrated with ***Mathematica***. For an helium nucleus moving around our previous pulsar, with a modest field strength $B_{surf} \sim 100$ T, we get :

$$\alpha_{grav} \approx 0.207, \qquad \alpha_{\mathrm{EM}} \approx 1.60 \times 10^{5}.$$

For a short period of time, the gravitational force is thus negligible and shouldn't have any noticeable effect on the trajectory. Of course, this may not be true over a long period of time and we should also take into account the fact that the magnetic field decreases much faster with distance than gravity ($B \sim r^{-3}$ and $g \sim r^{-2}$). For a rotation period of one second, we get $k_{*} \approx 2.10 \times 10^{-4}$. The Coriolis and centrifugal forces shouldn't have any noticeable effect close to the pulsar. This may easily be checked while doing the numerical integration, by turning ON and OFF the various force components. The figure below shows an example, from the parameters given above. The graduations on the horizontal cartesian axis are defined in normalised units (one graduation mark equals 100 km). The particle is moving with an initial velocity of 25% light speed.

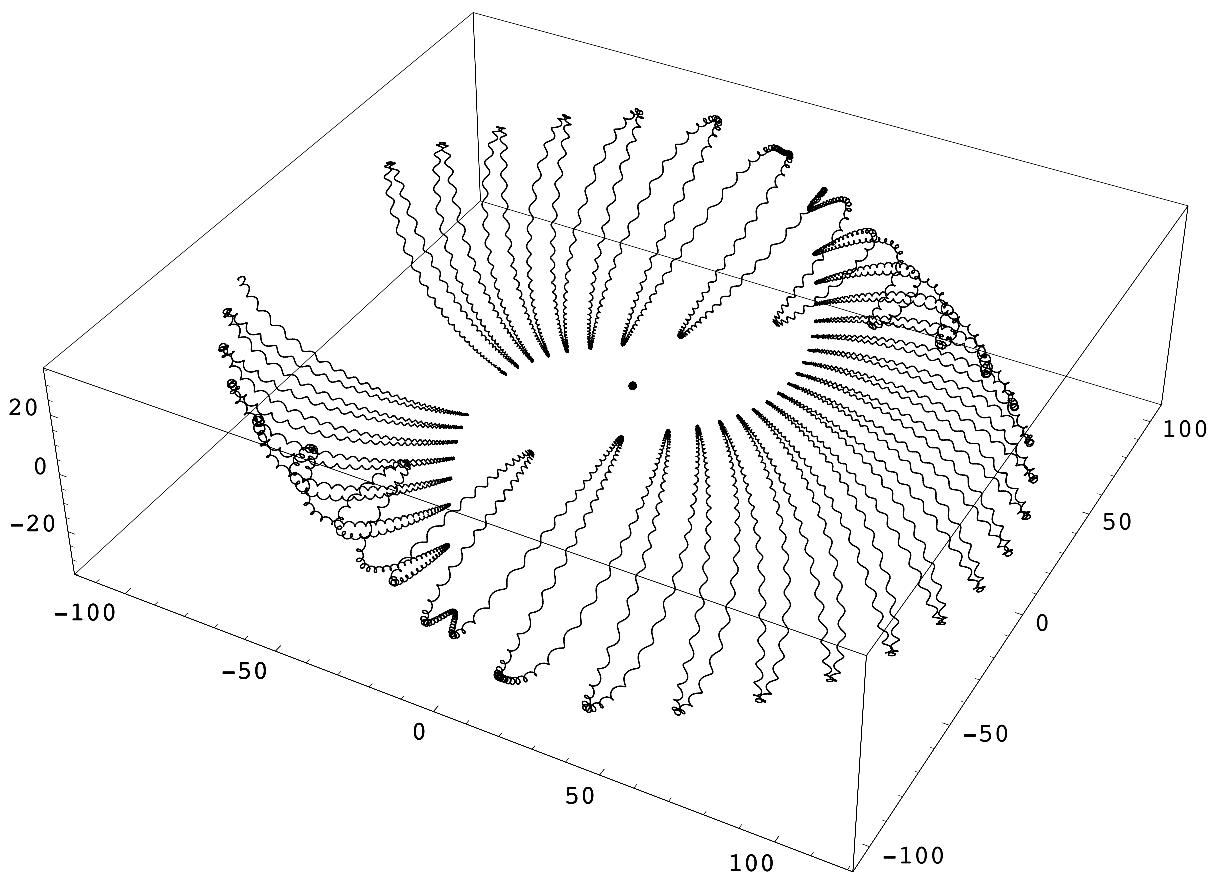

Figure 1 - Trajectory of an Helium nucleus

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